Announcements

1) Math Career Talks

Monday 1/28

2:30 CB 1030

Recall: A subset $S \subseteq \mathbb{R}$ has (Lebesgue) measure zero if $\forall\, \varepsilon > 0$, $\exists$ a countable collection $\{O_i\}_{i=1}^{\infty}$ of open intervals with

$$S \subseteq \bigcup_{i=1}^{\infty} O_i \text{ and}$$

$$\sum_{i=1}^{\infty} \ell(O_i) < \varepsilon .$$

A function $f: [a,b] \to \mathbb{R}$ is α-continuous for some $\alpha \geq 0$ in $\mathbb{R}$ and at $x \in [a,b]$ if $\exists\, \delta > 0$ such that $\forall\, y, z \in (x-\delta, x+\delta)$

$$|f(z) - f(y)| < \alpha.$$

f is uniformly α-continuous on $S \subseteq [a,b]$ if the same δ works for all $x \in S$.

Recall also

$D = \{x \in [a,b] \mid f \text{ is discontinuous at } x\}$

$D_\alpha = \{x \in [a,b] \mid f \text{ is not } \alpha\text{-continuous at } x\}$

We showed

$$D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$$

and D_α is closed $\forall$

$\alpha \geq 0$.

Theorem: (Lebesgue)

Let $f: [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ if and only if D has measure zero.

proof:

$\underline{\Longleftarrow}$ Let $M = \sup_{x \in [a,b]} |f(x)|$

Let $\varepsilon > 0$, $\alpha = \dfrac{\varepsilon}{4(b-a)}$

<u>Step 1:</u> $\exists$ disjoint open intervals $G_1, G_2, \ldots, G_k$ with $D_\alpha \subseteq \bigcup_{i=1}^{k} G_i \subseteq [a,b]$ and

$$\sum_{i=1}^{k} \ell(G_i) < \frac{\varepsilon}{4M}$$

We are assuming D has measure zero. Hence, since

$$D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}},$$

we know $D_{\frac{1}{n}}$ has measure zero $\forall\, n \in \mathbb{N}$ (any subset of a measure-zero set has measure zero).

If $\alpha > 0$, $\exists\, n \in \mathbb{N}$ with

$$\frac{1}{n} < \alpha \Rightarrow D_\alpha \subseteq D_{\frac{1}{n}}.$$

This implies D_α has measure zero for all $\alpha > 0$. In particular, for $\alpha = \frac{\varepsilon}{4(b-a)}$, D_α has measure zero. For this choice of α, let $\{O_i\}_{i=1}^{\infty}$ be open intervals with

$$D_\alpha \subseteq \bigcup_{i=1}^{\infty} O_i \text{ and}$$

$$\sum_{i=1}^{\infty} \ell(O_i) < \frac{\varepsilon}{4M}.$$

Since D_α is closed and $[a,b]$ is compact, $D_\alpha \subseteq [a,b]$ is compact.

Therefore, $\exists\ i_1, i_2, \ldots, i_n$ with $D_\alpha \subseteq \bigcup_{j=1}^{n} O_{i_j}$.

$\left(O_{i_j} \in \{O_i\}_{i=1}^{\infty} \ \forall\ 1 \leq j \leq n \right)$.

Let $y_1, y_2, \ldots, y_{2n}$ be the endpoints of $\{O_{i_j}\}_{j=1}^{n}$

By removing duplicates if necessary, we obtain

$$x_1 < x_2 < \cdots < x_k$$

where $x_i \in \{y_j\}_{j=1}^{2n}$

Let $G_i = (x_i, x_{i+1})$.

Note that if $D_\alpha \cap \{x_i\}_{i=1}^{k}$ is empty, then

$$D_\alpha \subseteq \bigcup_{i=1}^{k} G_i \quad \text{and}$$

$$\sum_{i=1}^{k} \ell(G_i)$$

$$\leq \sum_{j=1}^{n} \ell(O_{i_j})$$

$$\leq \sum_{i=1}^{\infty} \ell(O_i) < \frac{\varepsilon}{4M}.$$

By construction, $G_i \cap G_j = \{\phi\}$

$\forall \; i \neq j$.

If there is some intersection, we instead choose

$\sum \ell(O_i) < \frac{\varepsilon}{8M}$ and

since $\{x_1, x_2, \ldots, x_k\}$ is finite and hence of measure zero, we make open intervals containing each $x_i \in D_\alpha$ with endpoints not in D_α and repeat the construction of G_i's.

Step2: Let $K = [a,b] \setminus \bigcup_{i=1}^{k} G_i$.

Then f is uniformly α-continuous on K.

Since $D_\alpha \subseteq \bigcup_{i=1}^{k} G_i$, we have that if $x \in K$, then f is α-continuous at x.

The complement $\left(\bigcup_{i=1}^{k} G_i\right)^c$ is a closed set, hence

$$K = [a,b] \setminus \bigcup_{i=1}^{k} G_i$$

$$= [a,b] \cap \left(\bigcup_{i=1}^{k} G_i \right)^c$$

is a closed subset of $[a,b]$, hence compact. Therefore, since f is α-continuous $\forall$ $x \in K$, f is uniformly α-continuous on K.

Step 3: Choosing a partition

Since f is uniformly α-continuous on K, $\exists$ $\delta > 0$ such that $\forall x \in K$ and $\forall$ $y, z \in (x-\delta, x+\delta)$,

$$|f(y) - f(z)| < \alpha = \frac{\varepsilon}{4(b-a)}.$$

Choose $m \in \mathbb{N}$, $\frac{b-a}{m} < \delta$.

Let $n = \max\{m, 2k\}$. Then $\frac{b-a}{n} < \delta$.

Now consider the points $\{x_i\}_{i=1}^{2k}$ which are the endpoints of the disjoint open intervals G_i, $1 \le i \le k$.

If $x_1 < a$, remove x_1 from $\{x_i\}_{i=1}^{2k}$. If $x_{2k} > b$, remove x_{2k} from $\{x_i\}_{i=1}^{2k}$. Call the new set $\{y_i\}_{i=1}^{r}$ $2k-2 \le m \le 2k$.

Let P be the partition

$$\{y_i\}_{i=1}^{r} \cup \left\{a + \frac{j(b-a)}{n}\right\}_{j=0}^{n}$$

Suppose $P = \{z_i\}_{i=1}^{t}$ with

$$a = z_1 < z_2 < z_3 < \cdots < z_t = b.$$

$$U(f,P) - L(f,P) = \sum_{i=1}^{t-1} (M_i - m_i)(z_{i+1} - z_i)$$

$$= \sum_{i \in S} (M_i - m_i)(z_{i+1} - z_i) + \sum_{i \in S^c} (M_i - m_i)(z_{i+1} - z_i)$$

where $S = \left\{ i \mid (z_i, z_{i+1}) \subseteq \bigcup_{j=1}^{k} G_j \right\}$

Now $\sum_{i \in S} (M_i - m_i)(z_{i+1} - z_i)$

$\leq \sum_{i \in S} 2M (z_{i+1} - z_i)$

(Since $M = \sup_{x \in [a,b]} |f(x)|$

$\Rightarrow -M \leq m_i$ and $M \geq M_i$ $\forall i \in S$)

$\leq 2M \sum_{i=1}^{k} \ell(G_i) \leq 2M \cdot \frac{\varepsilon}{4M}$

$= \frac{\varepsilon}{2}$

Finally, if (z_i, z_{i+1}) is not contained in $\bigcup_{j=1}^{k} G_j$, then since the G_j's are disjoint intervals, $[z_i, z_{i+1}] \subseteq K$.

Then $M_i - m_i \leq \alpha \quad \forall\, i \in S^c$.

$$\Rightarrow \sum_{i \in S^c} (M_i - m_i)(z_{i+1} - z_i)$$

$$\leq \alpha \sum_{i \in S^c} (z_{i+1} - z_i)$$

$$\leq \alpha \left(\frac{b-a}{n}\right) |S^c|$$

Now $|S^c| \leq n + r$

$\leq 2n$

$$\Rightarrow \alpha \left(\frac{b-a}{n}\right) |S^c|$$

$$\leq 2\alpha(b-a)$$

$$= 2\left(\frac{\varepsilon}{4(b-a)}\right)(b-a) = \frac{\varepsilon}{2}$$

$$\Rightarrow U(f,P) - L(f,P)$$

$$= \sum_{i \in S} (M_i - m_i)(z_{i+1} - z_i) + \sum_{i \in S^c} (M_i - m_i)(z_{i+1} - z_i)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This implies that

f is integrable on $[a,b]$.

$\Rightarrow$ Friday!